Modern Type Theories and Their Applications [现代类型论及其应用]

Zhaohui Luo [罗朝晖]
Royal Holloway, Univ of London
Zhaohui.Luo@hotmail.co.uk
https://www.cs.rhul.ac.uk/home/zhaohui/

This talk – two parts

I. Modern Type Theories: brief introduction [现代类型论]
- Basics of MTTs
- Meta-theory and meaning theory
- Application in proof assistants based on MTTs

II. Two applications of MTTs:
- Univalent foundations & homotopy type theory [单价基础与同伦类型论]
- Formal semantics in MTTs (MTT-semantics) [现代类型论语义学]

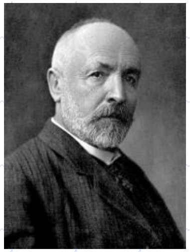
Studying type theory, I've collaborated with many people (only mentioning a few):
- Adams, Callaghan, Goguen, Pollack (type theory & proof assistants)
- Soloviev, Xue, Y. Luo, Lungu (coercive subtyping)
- Asher, Chatzikyriakidis, Maclean, Shi (MTT-semantics)

Part I. Modern Type Theories

[现代类型论]

Origin of type theory

- Foundations of mathematics and paradoxes
  - Naïve set theory (Cantor, …)
  - Paradox in naïve set theory (Russell 1903) [next slide]
  - Crisis in foundations of mathematics
- Set theory by Zermelo
  - Axiomatic set theory (1908; later ZFC etc.)
  - Widely accepted foundations in math community
- Type theory by Russell
  - Ramified type theory (*Principia Math.* 1910-13, 1925)
  - Vicious circle principle ("impredicativity" like $\forall X.X$)
  - Ramified hierarchy – problematic "axiom of reducibility"



Russell's paradox in naïve set theory

- Naïve concept of set with unrestricted comprehension:

  $\{ x \mid P(x) \}$ for any predicate P in FOL

- Russell's paradoxical set would exist if we accepted this:

  $R = \{ x \mid x \notin x \}$

- Then, by definition, we would have an absurd equivalence:

  $R \in R \Leftrightarrow R \notin R$ (*)

- BTW, R exists ➔ (*) ➔ logical inconsistency.

Simple type theory

- Ramsey (1926)
  - Logical v.s. semantic paradoxes
  - Russell's paradox v.s. (e.g.) Liar's paradox
  - Impredicativity is circular, but not vicious
  - So, Russell's ramified TT can be "simplified" to simple TT.

- Church's simple type theory (1940)
  - Formal system based on λ-calculus
  - Types as in ramified TT (e, t, e$\rightarrow$t, …)
  - Higher-order logic (formulas like $\forall X.X$)
  - Wide applications (Montague semantics, proof assistants, …)

Note: "Simple" could have another meaning: only "simple" types …

Modern Type Theories

- Martin-Löf has introduced/employed
  - Judgements, contexts, definitional equality
  - Dependent/inductive types, type universes
  - Curry-Howard principle of propositions-as-types

- Examples of MTTs [& implementing proof assistants]:
  - Predicative TTs:
    - MLTT – Martin-Löf's type theory [1975]; Agda
  - Impredicative TTs:
    - CC [Coquand & Huet 1988] and pCIC; Coq/Lean
    - UTT [Luo 1990, 1994]; Lego/Plastic

UTT [Luo 89, 94] – an example MTT

- UTT – Unifying theory of Dependent Types (MLTT + CC)

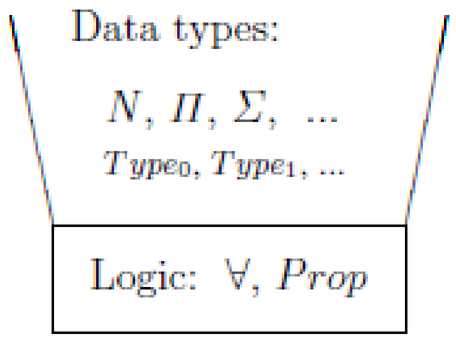
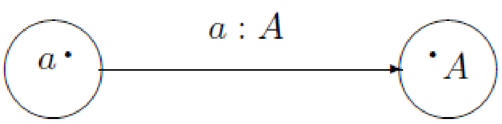


Fig. 1. The type structure in UTT.

- UTT has nice meta-theoretic properties
  - Goguen's PhD thesis on "Typed Operational Semantics" (1994)
  - Strong normalisation, which implies, e.g., consistency etc.

Judgements – basic notion in type theory



- Membership judgement
  - a : A – a is an object of type A.
- What is A? A can be: [see next slide]
  - data type: eg, Nat, A→B
  - propositional type: eg, ∀x:A.P
  - type universe: a type of some other types
- Comparison with set theory: [see slide]
  - Judgement "a : A" is <u>not</u> a logical formula
  - Different from "s ∈ S", which is a formula (say in FOL)
  - Logic is only a part of type theory (propositional types).

Π-types and ∀-props: examples of dependent types

- Πx:A.B(x) – dependent function type
  - Type for collection

    { f ∈ A→$\bigcup_{a\in A}$B(a) | ∀a∈A. f(a)∈B(a) }
  - f : Πx:Human.Parent(x)

    ➔ f(h) is father/mother of h (not others!)

$$\frac{\Gamma \vdash A\ type \quad \Gamma,\ x{:}A \vdash B\ type}{\Gamma \vdash \Pi x{:}A.B\ type}$$

$$\frac{\Gamma,\ x{:}A \vdash b : B}{\Gamma \vdash \lambda x{:}A.b : \Pi x{:}A.B}$$

$$\frac{\Gamma \vdash f : \Pi x{:}A.B \quad \Gamma \vdash a : A}{\Gamma \vdash f(a) : [a/x]B}$$

$$\frac{\Gamma,\ x{:}A \vdash b : B \quad \Gamma \vdash a : A}{\Gamma \vdash (\lambda x{:}A.b)(a) = [a/x]b : [a/x]B}$$

- Universal quantification

$$\frac{\Gamma \vdash A\ type \quad \Gamma,\ x{:}A \vdash P : Prop}{\Gamma \vdash \forall x{:}A.B : Prop}$$

  - Prop, the collection of propositions, is a type itself [impredicative universe with “circular” props like ∀X:Prop.X]
  - Propositions-as-types: propositions are (some) types [So, logic(s) is only a part of type theory – see next slide.]

Relationship between logic and set/type theory

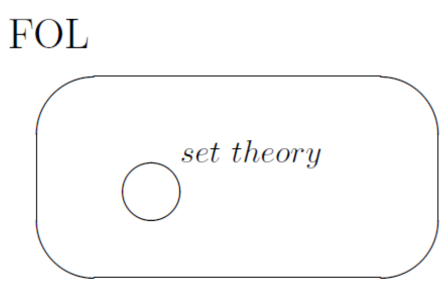


Figure 1: Set theory – a theory in first-order logic

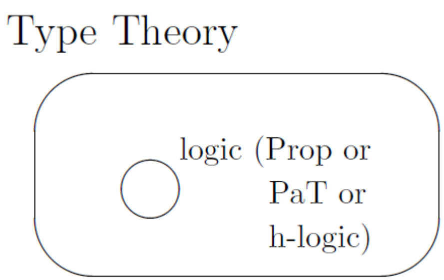


Figure 2: Logic is a part of type theory

Inductive types: an example

- Peano axioms: logical theory for natural numbers.
  [N is a predicate and n∈N stands for N(n)]

$$
\begin{array}{ll}
(P1) & 0 \in N \\
(P2) & \forall x.\ x \in N \Rightarrow succ(x) \in N \\
(P3) & \forall x, y.\ x, y \in N \wedge succ(x) = succ(y) \Rightarrow x = y \\
(P4) & \forall x.\ x \in N \Rightarrow 0 \neq succ(x) \\
(P5) & \forall P.\ P(0) \wedge [\forall x.\ x \in N \wedge P(x) \Rightarrow P(succ(x))] \Rightarrow \forall z.\ z \in N \Rightarrow P(z)
\end{array}
$$

- Martin-Löf's idea
  - Inductive types as "computational theories"
  - Example – Nat, the type of natural numbers

Rules for Nat

- Formation and introduction rules

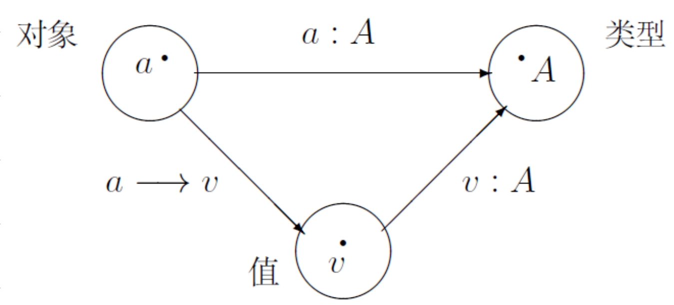
$$\frac{}{Nat\ type} \qquad \frac{}{0 : Nat} \qquad \frac{n : Nat}{succ(n) : Nat}$$

- Elimination rule

$$\frac{\Gamma, z : Nat \vdash C(z)\ type \quad \Gamma \vdash n : Nat \qquad \Gamma \vdash c : C(0) \quad \Gamma, x : Nat, y : C(x) \vdash f(x,y) : C(succ(x))}{\Gamma \vdash \mathcal{E}_{Nat}(c, f, n) : C(n)}$$

- Notes:
  - Introduction rules specify canonical objects [规范对象].
  - Elimination rule is Nat-induction + primitive recursion.
  - All Peano axioms are either rules or provable.

Meaning explanation

- Understanding based on computation:
  - 

    Example: A = Nat, a = 3+4, v = 7.
- How to guarantee that computation a→v terminates !?
  - Meta-theoretic study (eg, strong normalisation of UTT)
  - Meaning-theoretic argument (harmony of intro/elim rules)

Meta-theory

- Meta-theory of type theories
  - Computation is central.
    - Strong normalisation: All computations terminate.
    - This usually implies canonicity and logical consistency.
  - Sophisticated, tedious and rather hard to do
    - Many many theorems/lemmas/concepts/… [examples in next 2 slides]
  - ECC/UTT's meta-theoretic studies [Luo 1990, Goguen 1994]
- Caveat:
  - Meta-theory depends on consistency of meta-language (set theory) – believed to be true, but …
  - Desire/wish: can we argue for "correctness" directly?

Meta-theoretic theorems: examples

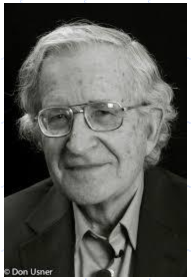
- Church-Rosser theorem (CR) [CR定理]
  - If a=b : A, then there exists c : A s.t. a → c and b → c.
- Subject Reduction (SR) [主题归约]
  - If a : A and a → b, then b : A.
- Strong Normalisation (SN) [强正规化]
  - Every computation from a well-typed term terminates.
- Logical consistency (in UTT) [逻辑相容性]
  - ∀X:Prop.X (false) is not provable (in the empty context).
- Decidability (of type-checking) [（类型检测的）可判定性]
  - It is decidable whether a judgement is correct (derivable).

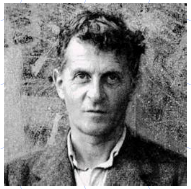
Example proof: logical consistency

- Proof (of consistency)
  - Assume that M : ∀X:Prop.X.
  - By SN & SR, we may assume that M is in normal form [范式].
  - So, M ≡ λX:Prop.M′ s.t. X:Prop |- M′ : X for some $M' \equiv M_1 \ldots M_n X$ (or other forms of "base term").
  - But we can then show that this would imply either Prop = X or Prop = Qx:A.B which, by CR, is impossible.
  - Therefore, M does not exist (∀X:Prop.X is not provable).

Theories of meaning

- Meaning is reference ("referential theory")
  - Word meanings are (abstract/concrete) objects.
  - c.f., platonism: Frege, …
- Meaning is concept ("internalist theory")
  - Word meanings are ideas in the mind.
  - c.f., Aristotle, Chomsky, …
- Meaning is use ("use theory")
  - Word meanings are understood by its uses.
  - c.f., Wittgenstein, …





Proof-theoretic semantics – use theory for logics


- Proof-theoretic semantics
  - Use theory for logical systems
  - Dummett, Prawitz, …
- Ideas
  - Pre-mathematical justification of logical rules (informally from "first principles", not meta-theoretically)
  - For logic: two aspects of use – verification and consequence
  - Harmony: intro/elim rules should be harmonious.
- Proof-theoretic semantics for type theories
  - Martin-Löf's meaning explanations (1984)
  - Type theory potentially has PTS, while set theory does not.
  - Current investigations: hypothetical judgements, impredicativity, …

Proof technology based on type theories

- Proof assistants – interactive proof development
  - MTT-based: Agda, Coq, Lean, Lego, NuPRL, Plastic, …
  - HOL-based: HOL, Isabelle, …
- Applications of proof assistants
  - Formalisation of mathematics
    - 4-colour theorem (Coq), Kepler conjecture (Isabelle)
    - Univalent foundations of mathematics

  - Computer Science:
    - program verification and advanced programming
  - Computational Linguistics
    - NL reasoning based on MTT-semantics (Coq)

Part II. Two applications of MTTs

- Univalent foundations & homotopy type theory
  [单价基础与同伦类型论]
- Formal semantics in MTTs (MTT-semantics)
  [现代类型论语义学]

Part II(1)

Univalent foundations of mathematics

[数学的单价基础]

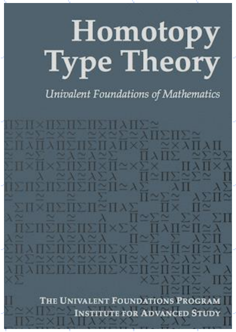
Univalent Foundations – alternative to set theory

- Vladimir Voevodsky (1966–2017)
  - Russian mathematician; Fields medalist (2002);
    Professor at Inst of Advanced Study, Princeton, USA
  - Worked on UF since 2005 (homotopy lambda calculus), developed UF library in Coq from 2010.
    - V. Voevodsky. An experimental lib of formalized math based on UF. MSCS, 2015.

- Voevodsky's key motivations and ideas
  - Proof-checking – we need foundations that make it possible.
    - Errors in his own papers, only discovered/confirmed 15/20 yrs later …
  - Groupoid [群胚] conception for higher dimensional math.
    - Groupoids, rather than categories, are "sets in the next dimension".
  - H-levels (homotopy levels of n-types) [Voevodsky 2009]
    - Propositions, sets, groupoids, …

Homotopy type theory (HoTT 2013)

- Development of HoTT
  - Formalisation of univalent foundations
  - Special year on univalent foundations of math.
    - 2012-13 at Inst of Advanced Study, Princeton, USA.
- HoTT = MLTT + UA + HITs
  - UA – univalence axiom [单价公理]
  - HITs – higher inductive types [高等归纳类型]



Univalence

- Univalence axiom (≃/Id for equivalence/identity of types):

  (UA) $\quad Id(A,B) \simeq (A \simeq B)$

  - Mathematical structuralism (invariance under equivalence)
    - UA is "unusual" (AxB ≃ BxA – they have same <u>expressible</u> properties.)
  - UA implies extensionality, both functional and propositional.
    - Note: Mathematics is extensional!
    - HoTT v.s. Extensional TT [Martin-Löf 1984] (ETT is problematic)
- UA as an <u>axiom</u> (in HoTT)?
  - "Axioms" are problematic in type theory!
  - With axioms, canonicity fails to hold.
    - Some "natural numbers" don't compute to canonical ones …
    - Correctness/adequacy of the foundational language is in doubt …!

Cubical type theory (Coquand et al, TYPES15, LICS18, …)

- Cubical type theory [立方类型论]
  - Research started in 2012-13 at Princeton, by Coquand et al, when Voevodsky had the conjecture: canonicity holds.
- Univalence is a theorem in the cubical type theory.
  - Canonicity for nats holds – a big step forward!
  - Normalisation and decidability? (to be proved)
- Experimental implementation in Agda-Cubical

Q: Is the cubical type theory the correct solution?

Higher inductive types

- Basic idea of HITs:
  - Ordinary induction is only about "points" (eg, 0 & succ(n)).
  - Higher induction extends it to "equalities/paths".
- Quotient types "A/R" – typical example (with ad hoc notation =)

  $|\_| : A \to A/R$

  $\forall x, y{:}A.\ R(x, y) \to |x|=|y|$
  - Quotient types were problematic ("setoid hell") – so real progress!
  - Current implementation (eg, Agda-cubical) still a bit cumbersome.
- Notes: Several research topics, including:
  - General schemata for HITs (still unknown)
  - Independent understanding of HITs

Direct v.s. indirect formalisations (side remark)

- Type theory is more effective (much more) when built-in entities are used directly.
- Application examples:
  - Formalisation of mathematics
    - HoTT-based proof development (e.g., HITs for quotients) [HoTT 2013]
    - In contrast with, e.g., setoids and related proofs (cumbersome …)
  - Program verification
    - Built-in functions as FP programs (and their verification)
    - In contrast with, e.g., "deep embedding + semantics" (cumbersome …)
  - Linguistic semantics
    - CNs-as-types in MTT-semantics (see below)
    - In contrast with, e.g., CNs-as-predicates in Montague semantics.

Part II(2). MTT-semantics

[现代类型论语义学]

Type-Theoretical Semantics

- Montague semantics (Montague 1930–1971)
  - MG: formal natural language semantics in set theory
  - Dominating in linguistic semantics since 1970s
  - Set-theoretic, using simple type theory as intermediate
- MTT-semantics: formal semantics in modern type theories
  - Ranta (1994): formal semantics in Martin-Löf's type theory
  - Recent study on MTT-semantics → full-scale alternative to MG
    - Z. Luo. Formal Semantics in MTTs with Coercive Subtyping. L&P 35(6). 2012.
    - S. Chatzikyriakidis and Z. Luo. Formal Semantics in MTTs. Wiley, 2020.
  - Research context on rich typing in NL (many researchers …)
    - S. Chatzikyriakidis and Z. Luo (eds.) Modern Perspectives in Type Theoretical Semantics. Springer, 2017.

MTT-semantics: basic categories

Category	**Semantic Type**
S	Prop (the type of all propositions)
CNs (book, human, …)	types (each common noun is interpreted as a type)
IV	$A \to Prop$ (A is the "meaningful domain" of a verb)
Adj	$A \to Prop$ (A is the "meaningful domain" of an adjective)
Adv	$\Pi A{:}CN.(A \to Prop) \to (A \to Prop)$ (polymorphic on CNs)

Simple example: [John talks] = talk(j) : Prop
where j : Human and talk : Human→Prop.

(*) In MTT-semantics, common nouns (CNs) are types rather than predicates as in Montague semantics.

Modelling Adjectival Modification: Case Study

Classical classification	Example	Characterisation	MTT-semantics
intersective	handsome man	Adj(N) ➔ N & Adj	$\Sigma x{:}Man.handsome(x)$
subsective	large mouse	Adj(N) ➔ N (Adj depends on N)	large : $\Pi A{:}CN.\ A \rightarrow Prop$ large(mouse) : $Mouse \rightarrow Prop$
privative	fake gun	Adj(N) ➔ ¬N	$G = G_R + G_F$ with $G_R \leq_{inl} G$, $G_F \leq_{inr} G$
non-committal	alleged criminal	Adj(N) ➔ nothing	$H_{h,Adj} : Prop \rightarrow Prop$

- [Chatzikyriakidis & Luo 13, 17 & 20; Luo, Shi & Xue 22]

Note on Subtyping in MTT-semantics

- Simple example
  - A human talks. Paul is a handsome man. Does Paul talk?
  - Semantically, can we type talk(p)?
    - talk : Human→Prop and p : [handsome man]
  - Yes, because p : [handsome man] $\leq$ Man $\leq$ Human
- Subtyping is crucial for MTT-semantics
  - Coercive subtyping [Luo 1999, Luo, Soloviev & Xue 2012] is adequate for MTTs and we use it in MTT-semantics.
  - Note: Traditional subsumptive subtyping is inadequate for MTTs
    - Canonicity fails with subsumptive subtyping.

Advanced features in MTT-semantics: examples

- Copredication and dot-types [Luo 09, XL 12, CL 18]
- Linguistic coercions via coercive subtyping [Asher & Luo 12]
- Signatures for linguistic contexts [Luo 14, Lungu & Luo 16]
- MTT event sem. (dependent event types) [Luo & Soloviev 17]
- Propositional forms of judgemental inter. [Xue et al 18, 23)]
- **MTT-semantics in $\mathbf{MLTT_h}$ [Luo (LACompLing 2018)] (*)**
- CNs as setoids [CL 18] (and CNs as HITs – in progress)
- Dependent categorial grammar [Luo 24]

(*) MTT-semantics in a predicative type theory? – next two slides.

MTT-semantics in Martin-Löf's TT – a problem

- Martin-Löf's type theory in formal semantics
  - Munnick, Sundholm, Ranta & many others
  - All use PaT logic – propositions as types.
  - But Martin-Löf goes one step further: types = propositions!
  - This is where the problem arises [Luo (LACL 2012)].
- Example: a handsome man is (m,p) : Σx:Man.handsome(x)
  - Two handsome men are the same iff they are the same man (and how to prove they are handsome should be irrelevant!)
  - Proof irrelevance (any two proofs of the same proposition are the same.)
  - But, in MLTT with PaT logic, this would mean every type collapses! Absurd.
- So, MLTT with PaT logic is <u>inadequate</u> for MTT-semantics.
  - Developing MTT-semantics in UTT is OK where proof irrelevance is possible.

MLTT$_h$: Extension of MLTT with H-logic

- H-logic ("H" for h-levels due to Voevodsky)
  - A proposition is a type with at most one object.
  - Logical operators (examples):
    - $P \supset Q = P \to Q$ and $\forall x{:}A.P = \prod x{:}A.P$
    - $P \vee Q = |P+Q|$ and $\exists x{:}A.P = |\Sigma x{:}A.P|$

    where $|A|$ is propositional truncation (a form of HITs).
- MLTT$_h$ = MLTT + h-logic (subsystem of HoTT) [Luo 2019]
  - Proof irrelevance is "built-in" in h-logic (by definition).
  - Note: MLTT$_h$ is a proper extension of MLTT.
  - Claim: MLTT$_h$ is adequate for MTT-semantics.

Research monograph on MTTs in Chinese

罗朝晖：现代类型论的发展与应用。
清华大学出版社，2024年。

Z. Luo. Modern Type Theories: Their Development and Applications. Tsinghua Univ Press, 2024. (In Chinese)

网址：http://www.tup.tsinghua.edu.cn/booksCenter/book_09109701.html